Tunneling as a stochastic process: A path-integral model for microwave experiments

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Delay time results obtained in microwave experiments at frequencies above and below the cutoff frequency of different waveguide sections are interpreted on the basis of wave propagation in the presence of dissipative effects. Kac's original suggestion was the starting point for the formulation of a stochastic model, which has now been substantially improved, also in relation to the transition-elements theory of Feynman-Hibbs. In this way, an approach to the problem is provided, which is completely distinct from the ones formulated elsewhere.

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Since the pioneering work by Kac, who gave a path-integral description of the telegrapher's equations and of the underlying Poisson random walk [1], there has been continuing interest in this kind of approach. DeWitt-Morette and Foong [2] provided a solution to the telegrapher's equations in terms of ordinary integrals, obtaining also the distributions of the first-passage time for the Poisson random walk [3]. In the meantime, a relation was established between quantum relativistic motions and the telegrapher's equations which , if analytically continued, results in the Dirac equation [4] and in the Klein-Gordon equation [5]. Indeed, along these lines, it was possible to derive a simplified model for tunneling (or traversal) time that accounts for the propagation of a pulse either above or below the cutoff frequency of a waveguide, which corresponds to a rectangular quantum-mechanical barrier [6].

The purpose of the present work is to report an improved version of a model, also based on the telegrapher's equation, that is capable of giving a reasonable description of the previously obtained and more recent experimental results, obtained in the microwave range. A demonstration of the stochastic nature of the process is given, which reinforces the previously made assumptions for which, further experimental evidence is now obtained.

The essential issue of the model of Ref. [6] can be summarized as follows. In the absence of dissipative effects, in a semiclassical approach the traversal time of a rectangular barrier can be expressed as $\tau_S = L/|v_g|$, where L is the length of the barrier and v_g the group velocity which, in the tunneling region of the spectrum (or below the cutoff frequency), is imaginary. Taking dissipation into account simply produces a shift of the peak of τ_S from $v_g=0$, or the cutoff frequency, towards the higher frequencies by an amount that depends on the dissipative parameter a, thus entering the telegrapher's equation, since this peak occurs at $v^2=a^2$.

To be more precise, the delay time is roughly given by $L/w_{1,2,3}$, where for $v^2>a^2$, $w_1=(v^2-a^2)^{1/2}$; and for $0<v^2<a^2$, $w_2=(a^2-v^2)^{1/2}$; for $v^2<0$, $w_3=(v^2+|v^2|)^{1/2}$. This assumption is improper, however, since quantity a must be homogeneous with $\omega=2\pi\nu$, ν being the frequency (rather than with v), that is, both a and ω have dimensions of (time)$^{-1}$ [7] [see Eq. (1) in Ref. [6]]. In a more refined treatment of the delay time, by considering a wave signal of the type $\sin(x-vt)$, we found essential confirmation of these results, except in the case of $0<v^2<a^2$, where the delay was found to be independent of the length L of the barrier, while in the other two cases ($v^2>a^2$ and $v^2<0$) the delay rightly depended on L [8]. For these reasons, the search for a more refined model seemed to be worthwhile.

According to Ref. [2], the solution of the telegrapher's equation can be expressed by the integral

$$F(x,t)=\int_{-\infty}^{\infty}[\alpha\phi(x,r)+\beta\phi(x,-r)]g(t,r)dr, \tag{1}$$

where $\phi(x,r)$ is a solution of the wave equation, without dissipation, and α and β are arbitrary mixing coefficients, so that $\alpha+\beta=1$. The boundary conditions are $F(x,0)=\phi(x,0)$ and $(\partial F/\partial t)_{t=0}=(\alpha-\beta)(\partial\phi/\partial t)_{t=0}$. The two-variable function $g(t,r)$, with $-t<r\leqslant t$, is the density distribution of a *randomized time* r (t is the normal time) which, tends asymptotically, for $t\gg r$, [disregarding a δ contribution of the type $e^{-at}\delta(t-r)$], to a Gaussian [9]

$$g(t,r)\rightarrow\sqrt{\frac{a}{2\pi t}}\exp\left(\frac{-ar^2}{2t}\right), \tag{2}$$

where a is the dissipative parameter entering the wave equation. Thus, the standard deviation of Eq. (2) is given simply by $\sigma=\sqrt{t/a}$. The exact expression of σ can be obtained as the square root of the variance reported by Eq. (50) in Ref. [8], namely,

$$\sigma=\left(\frac{t-\overline{r}}{a}-\overline{r}^2\right)^{1/2}=\left[\frac{t}{a}-\frac{(1-e^{-2at})(3-e^{-2at})}{4a^2}\right]^{1/2}, \tag{3}$$

where $\overline{r}=(1/2a)[1-\exp(-2at)]$; for $t\gg 1/a$, Eq. (3) tends to the simplified expression given above. The average time $\overline{r}$ has to be interpreted as the fictitious time it would take a

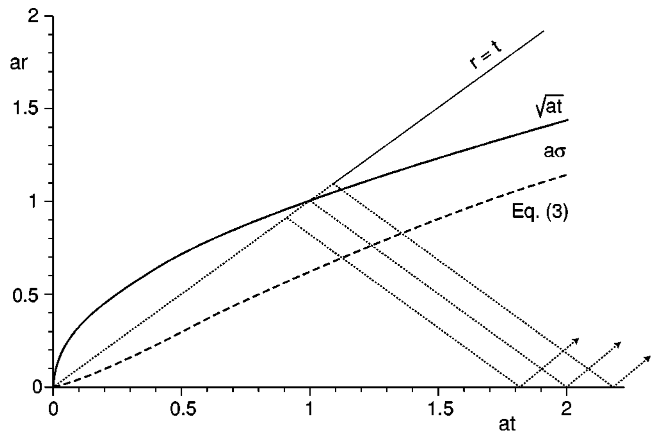

FIG. 1. The curve of the standard deviation σ multiplied by a ($a\sigma$) in the plane (ar,at) represents the border line of the half area containing the paths with a probability of $\sim 68\%$. The continuous line, given by $\sqrt{at}$, corresponds to the approximate distribution Eq. (2), while the dashed line is given by the exact expression (3). The straight line $r=t$ represents processes without reversals and the dotted lines represent typical paths with reversals.

particle to reach the average distance $L=v\overline{r}$ if it was always moving with velocity v, without reversals [1,2].

We can recall that Kac's work basically consisted in demonstrating that the telegrapher's equation is equivalent to the stochastic motion of a particle moving in a straight line, with constant velocity v, which suffers collisions that can reverse its velocity with probability $a\delta t$, after each step δx, and with continue in the same direction probability $1-a\delta t$. The resulting paths are, therefore, of a zigzag (or checkerboard) type in the space-time (x,t) plane consisting of segments with slope ± 1 for $v=1$ or, if we prefer, in the (r,t) plane for any v, since $x=vr$ [10]. Curve $\sigma(t)$ can be considered the borderline of the half area containing these paths. More precisely, the probability of the path being inside is $\sim 68\%$, if the distribution is of a Gaussian type (see Fig. 1). We now wish to estimate the typical extension of the path segments (or steps).

The first indication was given in Ref. [10] where, for relativistic particles of mass m, the time scale was found to be in Compton wavelengths over the light velocity c, namely, $\Delta t=\hbar/mc^2$. More precisely, in Ref. [11] it was estimated that the most probable step size is given simply by the above Δt. Within the electromagnetic framework, this quantity should become of the order of $1/\omega=1/2\pi\nu$. However, we know that the quantity $mc^2/\hbar$ is related to the dissipative parameter a (see below) when the connection is established between quantum relativistic motion (Dirac or Klein-Gordon equation) and the telegrapher's equation [4,5]. This appears to be the most plausible connection for an estimate of the extension of average steps. In fact, even in considering the distribution function $g(r,t)$ in its simplified Gaussian form (2), we find that the point of intersection of curve $\sigma(t)$ with the straight line $r=t$ [the line corresponding to $\delta(t-r)$, which represents processes without reversals] occurs for $r\equiv\sigma=a^{-1}$. In fact, if we take the distribution in its Gaussian approximation, for which, [see Eq. (2)] $r^2=\sigma^2=t/a$, for $r=t$ we obtain $r=a^{-1}$. It may be objected that this result holds true only for $t\gg a^{-1}$ while for smaller values of t the exact expression of σ given by Eq. (3), which supplies values sensibly smaller than the ones given by Eq. (2), should be adopted. However, in the first approximation, we can assume $\Delta r,\Delta t\simeq a^{-1}$ to be a reasonable measure of the most probable step size in time of the stochastic processes being considered. The step size in space will be given by $\Delta x=v(\Delta t,\Delta r)\simeq v/a$. This holds true both for classically allowed processes and for classically forbidden ones, provided that, according to the *ansatz* of Ref. [9], the roles of the variables t and r are exchanged when passing from classical to tunneling motions. This satisfies the intuitive result of a slowing down of the motion in the classical regime, while in the tunneling one we have just the opposite behavior. This means that while for classically allowed motion the effective space is $L=v\overline{r}$ and the true time is t, in the case of tunneling, the effective space will be $L=vt$ and the true time will be $\underline{r}$. Consequently, the traversal time for allowed processes is given by [9]

$$t_1=-\frac{1}{2a}\ln\left(1-2a\frac{L}{v}\right)\quad(2aL=\tilde{a}<v),\tag{4}$$

while for the forbidden, or tunneling, processe the said time will be given by [12]

$$t_3=\frac{1}{2a}(1-e^{-2aL/v})\quad(v^2<0).\tag{5}$$

Note that both expressions (4) and (5) rightly tend to L/v for $a\to 0$; the agreement with the simplified model of Ref. [6] is attained by taking $2aL=\tilde{a}$ (the dimension of which is velocity) as the measure of the dissipative effect.

For $0<v<\tilde{a}$, the form of the traversal time was not envisaged in Ref. [9]; however, we can try to adopt the same Eq. (4), properly continued to negative arguments, that is,

$$t_2=-\frac{1}{2a}\left[\ln\left|1-2a\frac{L}{v}\right|\pm i(2k+1)n\right]$$
$$(0<v<2aL=\tilde{a},\ k=0,1,2,\dots).\tag{6}$$

By retaining only the real part (the measured time is, of course, real), and making a further condition that such time must be positive and comparable, for sufficiently large a, with t_3, we have

$$t_2=-\frac{1}{2a}\ln\left(2a\frac{L}{v}-1\right)+\frac{f}{2v}\quad(0<v<2aL=\tilde{a}),\tag{7}$$

where f is an arbitrary numerical factor of the order of unity [13]. In Fig. 2 we show the three quantities t_1, t_2, and t_3 normalized to L, taking $2aL=\tilde{a}$ as an independent variable. The resulting curves represent a plausible candidate model for interpreting the experimental results. In fact, besides showing the typical peak around $\tilde{a}=v$, already predicted by the simplified model of Ref. [6], all the curves rightly depend

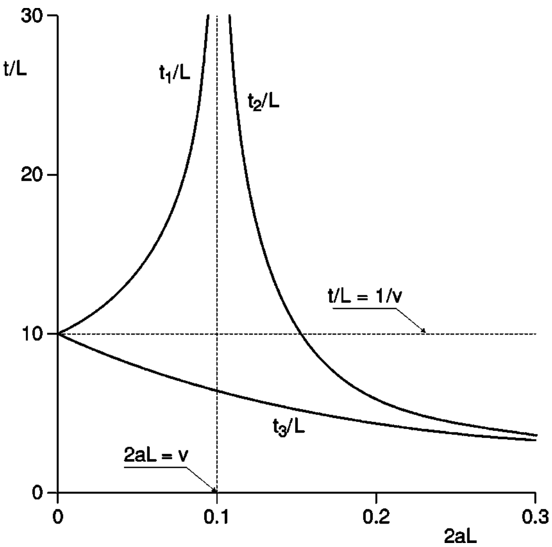

FIG. 2. Time delay as given by Eqs. (4), (5), and (7) for $v=0.1$, normalized to L, that is, $t_{1,2,3}/L$, as function of $\tilde{a}=2aL$.

on L. The inclusion of the corrective last term in Eq. (7) allows for an almost exact connection of t_2 with t_3 at sufficiently large values of $\tilde{a}$ or, as will be seen, for $v \rightarrow 0$. For this purpose, we must consider Eqs. (4), (5), and (7) as functions of v^2, or as functions of the frequency throughout the relation, which gives the square of the group velocity in a rectangular waveguide [14]

$$\frac{v_g^2}{c^2}=\left[1-\left(\frac{\nu_0}{\nu}\right)^2\right]\simeq 2\frac{(\nu-\nu_0)}{\nu_0}, \qquad (8)$$

where the last member holds true for frequency values comparable with the cutoff frequency ν_0.

We are now in a position to compare the experimental results with the aforementioned model. The experimental data are taken from Refs. [15,16]; in part they are new, but were obtained by using the same technique as that of Refs. [15,16], to which we refer for technical details. In Fig. 3, we report the measured values of delay obtained with three barrier lengths $L=10$, 15, and 20 cm, which correspond to as many waveguide sections with the same length and a cutoff frequency $\nu_0=9.495$ GHz. From the peak positions we can determine the values for the parameter $\tilde{a}/c$, throughout Eq. (8) rewritten as $a/c=(2\Delta\nu_0/\nu_0)^{1/2}$, where $\Delta\nu_0=\nu_p-\nu_0$ is the frequency shift of the peak position ν_p with respect to the cutoff frequency. This means that while in a pure semiclassical model the singularity (peak) is centered at $\nu=\nu_0$, in this modified model, the peak is shifted from ν_0 to an effective cutoff frequency ν_p. By expressing the multiplying factor $1/2a$ in Eqs. (4),(5), and (7) as $L/\tilde{a}$, we obtain the curves reported in Fig. 3, which give the first approximate description of the experimental data. The agreement is acceptable in the spectrum portion described by t_2 [Eq. (7)], while it is less satisfactory in the portions described by t_1 [Eq. (4)] and t_3 [Eq. (5)].

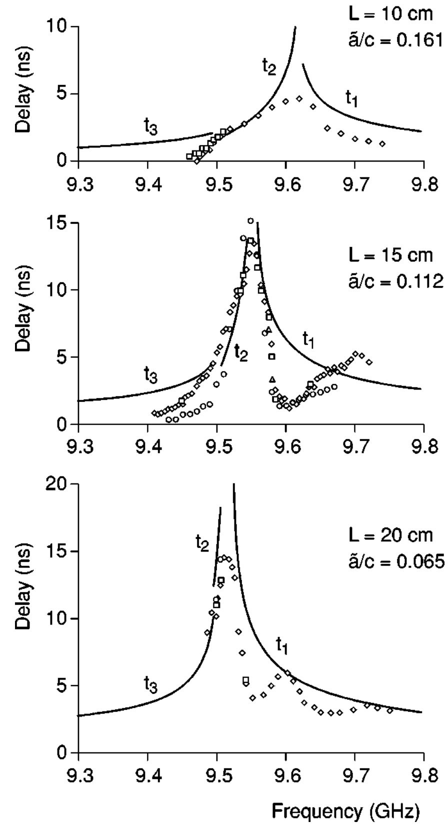

FIG. 3. Delay time results (as obtained in different series of measurements) relative to subcutoff waveguide sections with $L=10$, 15, and 20 cm. The cutoff frequency is $\nu_0=9.495$ GHz. The continuous lines are given by Eqs. (4), (5), and (7) multiplied by L/c, when considered as functions of the frequency, for selected values of the parameter $\tilde{a}/c$.

Before considering possible improvements to the model, we want to estimate the step extension of the stochastic motion. According to the analysis previously made, the typical time step will be evaluated as $\Delta t \simeq a^{-1}=(2L/c)/(\tilde{a}/c)$. However, by considering the exact expression of the standard deviation of the distribution $g(r,t)$, given by Eq. (3), we have a further reduction in the value $(1.2L/c)/(\tilde{a}/c)$. With the parameter values reported in Fig. 3 for the three cases considered, we obtain $\Delta t \simeq 2.5$, 5.4, and 12 ns, for $L=10$, 15, and 20 cm, respectively. As for the space steps, we have to multiply these values for the corresponding velocities which depend on the frequency. Taking, for instance, the values corresponding to the peaks of the spectral dependence of the delay time, for $\Delta x = v\Delta t$ we easily obtain the values $\Delta x \simeq 5$, 5.4, and 14.5 cm, for $L=10$, 15, and 20 cm, respectively. These values confirm the prediction of Ref. [11] regarding the persistence of the correlation in the direction of successive elementary steps: the grain of the process should be of the order of $1/\omega = 1/2\pi\nu$ which, for $\nu \simeq 10$ GHz, turns out to be $\delta t \simeq 16$ ps and δx of a few mm [17]. These considerations suggest a different way of interpreting the reported data in tunneling, as well as in the allowed region, in the neighborhood of a cutoff frequency which is still in the vicinity of the nominal cutoff one.

The zigzag behavior of the paths leads, for the case of tunneling, to a simple relation between Δx and Δt, expressed by Eq. (7.49) in Ref. [10]. It can be rewritten, by identifying c with v, as

$$\Delta t\langle 1\rangle = i\,\frac{mc^2}{\hbar}\left\langle \left(\frac{\Delta x}{v}\right)^2\right\rangle, \tag{9}$$

where $\langle 1\rangle$ is the propagator, and the symbol $\langle\cdots\rangle$ indicates that we are considering the transition element, a sort of average. Now, by assuming $\Delta t\langle 1\rangle = \langle t\rangle_R$ and $\Delta x = L$, and by identifying, according to the analytic-continuation prescription of Ref. [4], $imc^2/\hbar$ with a, Eq. (9) becomes [18]

$$\langle t\rangle_R = a\left\langle \left(\frac{L}{v}\right)^2\right\rangle, \tag{10}$$

where $\langle t\rangle_R$ represents the real part of the average time, a result which confirms the one reported in Ref. [17], although it was obtained in a different way.

As for the observed behavior of the experimental results in the allowed region, which is above a given cutoff frequency, its undulating shape suggests considering a different expression, also obtained within the framework of the transition elements of the path-integral theory [10]. According to the analysis of Refs. [19,20], where different situations of decaying waves were considered, the traversal time can be expressed by the approximate relation, derived from Eq. (7.69) in Ref. [10], as

$$\langle t\rangle_R \simeq \frac{L}{v}\left[1 - A\cos\left(2a\frac{L}{v}\right)\right]\langle 1\rangle, \tag{11}$$

where the propagator $\langle 1\rangle$ can be related to the attenuation of the waves, and A is a quantity depending on a and ω. In our case, we tried to use Eqs. (10) and (11) to fit the experimental data by taking a, A, and the cutoff frequency ν_0 as the adjustable parameters. As we shall see, the required values for a and ν_0 are not the same when going from classically allowed to classically forbidden processes; but this is not surprising, since there is no reason to believe that such quantities must be the same when the regime, in going from allowed to forbidden processes, is changed.

A more satisfactory description of the experimental data is obtained, as shown in the example of Fig. 4, where the curves are obtained from Eqs. (10) and (11), for plausible parameter values. Note that, in this case, the experimental data are fitted by only two expressions (rather than three, as in the previous formulation of the model), which can be also derived from the same basic formula [20,21].

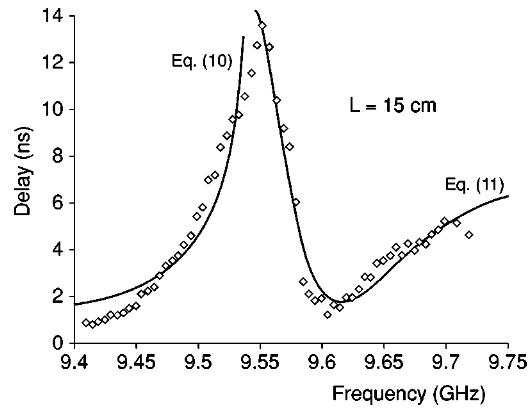

FIG. 4. For the case of $L = 15$ cm (the set of data refers to the most detailed example), the curves of delay are given by Eq. (10) with $a = 0.22$ (ns)$^{-1}$, below cutoff frequency $\nu_0 = 9.56$ GHz, and by Eq. (11), with $a = 0.98$ (ns)$^{-1}$, $A = 1.34$, above cutoff frequency $\nu_0 = 9.5$ GHz, a propagator value of ~ 1, and an offzero of ~ 2.8 ns.

It seems, therefore, that we can safely conclude that the stochastic models considered, especially that of the last version obtained according to the path-integral treatment (to which the stochastic motion can be reconducted as well), are capable of giving a satisfactory interpretation of the experiments carried out in the microwave range.

We are quite aware that other interpretations have been given (in the same region, as well as in other spectral regions) [22]. Among these, the best interpretational scheme remains, all things considered, the Hartman phase-time model [23]. This is sometimes referred to as the *Hartman effect*, when invoked in order to explain superluminal effects that can indeed be observed when the frequency is sufficiently below the cutoff one. This type of behavior gave rise, however, to some controversial interpretations, as can be found also in the recent literature [24,25]. Our approach remains completely distinct, based as it is on a hypothesis of stochasticity of motion (as a consequence of dissipation), according to Kac's early (and well-founded) suggestion on the nature of wave propagation with dissipation.

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